ATUL CLASSES

S.C.O.3 SOHI COMPLEX BALTANA(ZIRAKPUR), 8968103999

Test / Exam Name: Atul Classes | **Standard: 12th Science** | **Subject: Mathematics**

Student Name: ______________ **Section:** ______________ **Roll No.:** ______________

Questions: 20 | **Time: 01:00 hh:mm** | **Marks: 121**

Q1. Evaluate the following integrals: **7 Marks**

$\int \frac{1}{\cos x+\operatorname{cosec} x} dx$

Ans:

Let $I = \int \frac{1}{\cos x+\operatorname{cosec} x} dx$

$= \int \frac{\sin x}{1+\cos x \sin x} dx$

$= \int \frac{2\sin x}{2+2\cos x \sin x} dx$

$= \int \frac{\sin x+\cos x+\sin x-\cos x}{2+2\cos x \sin x} dx$

$= \int \frac{\sin x+\cos x}{2+2\cos x \sin x} dx + \int \frac{\sin x-\cos x}{2+2\cos x \sin x} dx$

$= \int \frac{\sin x+\cos x}{3-\sin^2 x-\cos^2 x+2\cos x \sin x} dx + \int \frac{\sin x-\cos x}{1+\sin^2 x+\cos^2 x+2\cos x \sin x} dx$

$= \int \frac{\sin x+\cos x}{3-(\sin x-\cos x)^2} dx + \int \frac{\sin x-\cos x}{1+(\sin x+\cos x)^2} dx$

$= I_1 + I_2 \ldots (1)$

Where, $I_1 = \int \frac{\sin x+\cos x}{3-(\sin x-\cos x^2)} dx$ and $I_2 = \int \frac{\sin x-\cos x}{1+(\sin x+\cos x)^2} dx$

Now,

$I_1 = \int \frac{\sin x+\cos x}{3-(\sin x-\cos x)^2} dx$

Let $(\sin x - \cos x) = t$

On differentiating both sides, we get

$(\cos x + \sin x)dx = dt$

$\therefore I_1 = \int \frac{1}{3-(t)^2} dt$

$= \frac{1}{2\sqrt{3}} \log \left| \frac{\sqrt{3}+t}{\sqrt{3}-t} \right| + C_1$

$= \frac{1}{2\sqrt{3}} \log \left| \frac{\sqrt{3}+\sin x-\cos x}{\sqrt{3}-(\sin x-\cos x)} \right| + C_1 \ldots (2)$

Now,

$I_2 = \int \frac{\sin x-\cos x}{1+(\sin x+\cos x)^2} dx$

Let $(\sin x + \cos x) = t$

On differentiating both sides, we get

$(\cos x - \sin x)dx = dt$

$\therefore I_2 = -\int \frac{1}{1+(t)^2} dt$

$= -\tan^{-1} t + c_2$

$= -\tan^{-1}(\sin x + \cos x) + c_2 \ldots (3)$

On substituting (2) and (3) in (1), we get

$I = \frac{1}{2\sqrt{3}} \log \left| \frac{\sqrt{3}+\sin x-\cos x}{\sqrt{3}-\sin x+\cos x} \right| - \tan^{-1}(\sin x + \cos x) + C$

Hence, $\int \frac{1}{\cos x+\operatorname{cosec} x} dx = \frac{1}{2\sqrt{3}} \log \left| \frac{\sqrt{3}+\sin x-\cos x}{\sqrt{3}-\sin x+\cos x} \right| - \tan^{-1}(\sin x + \cos x) + C$

Q2. Evaluate the following integrals: **6 Marks**

$\int \frac{1}{\sin x(3+2\cos x)} dx$

Ans:

Let $I = \int \frac{1}{\sin x(3+2\cos x)} dx$

$= \frac{\sin x \, dx}{\sin^2 x(3+2\cos x)}$

$= \frac{\sin x \, dx}{(1-\cos^2 x)(3+2x)}$

Let $\cos x = t$

$\Rightarrow -\sin x \, dx = dt$

$\therefore I = \int \frac{dt}{(t^2-1)(3+2t)}$

Now,

Let $\frac{1}{(t^2-1)(3+2t)} = \frac{A}{t-1} + \frac{B}{t+1} + \frac{C}{3+2t}$

⇒ 1 = A(t + 1)(3 + 2t) + B(t - 1)(3 + 2t) + C (t^2 - 1)

Put t = -1

⇒ 1 = -2B $\Rightarrow A = \frac{1}{10}$

Put $t = -\frac{3}{2}$

$\Rightarrow 1 = \frac{5}{4}C \Rightarrow C = \frac{4}{5}$

Thus,

$I = \frac{1}{10}\int \frac{dt}{t-1} - \frac{1}{2}\int \frac{dt}{t+1} + \frac{5}{4}\int \frac{dt}{3+2t}$

$= \frac{1}{10}\log|t-1| = \frac{1}{2}\log|t+1| + \frac{2}{5}\log|3+2t| + C$

Hence,

$I = \frac{1}{10}\log|\cos x - 1| - \frac{1}{2}\log|\cos x + 1| + \frac{2}{5}\log|3 + 2\cos x| + C$

Q3. Evaluate the following integrals: **6 Marks**

$\int_0^{\frac{\pi}{2}} \frac{1}{5+4\sin x}\,dx$

Ans:

Let $I = \int_0^{\frac{\pi}{2}} \frac{1}{5+4\sin x}\,dx$ Then,

$I = \int_0^{\frac{\pi}{2}} \frac{1}{5+4\left(\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)}\,dx$

$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{1+\tan^2\frac{x}{2}}{5\left(1+\tan^2\frac{x}{2}\right)+8\tan\frac{x}{2}}\,dx$

$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\sec^2\frac{x}{2}}{5\tan^2\frac{x}{2}+8\tan\frac{x}{2}+5}\,dx$

Let $\tan\frac{x}{2} = t$ Then, $\frac{1}{2}\sec^2\frac{x}{2}\,dx = dt$

When $x = 0, t = 0$ and $x = \frac{\pi}{2}, t = 1$

$\therefore\ I = 2\int_0^1 \frac{1}{5t^2+8t+5}\,dt$

$\Rightarrow I = 2\int_0^1 \frac{1}{(\sqrt{5}t)^2+8t+5+\left(\frac{4}{\sqrt{5}}\right)^2-\left(\frac{4}{\sqrt{5}}\right)^2}\,dt$

$\Rightarrow I = 2\int_0^1 \frac{1}{\left(\sqrt{5}t+\frac{4}{\sqrt{5}}\right)^2+\frac{9}{5}}\,dt$

$\Rightarrow I = \frac{2}{3}\left[\tan^{-1}\left(\frac{\sqrt{5}t+\frac{4}{\sqrt{5}}}{\frac{3}{\sqrt{5}}}\right)\right]_0^1$

$\Rightarrow I = \frac{2}{3}\left[\tan^{-1}3 - \tan^{-1}\frac{4}{3}\right]$

$\Rightarrow I = \frac{2}{3}\left[\tan^{-1}\left(\frac{3-\frac{4}{3}}{1+3\times\frac{4}{3}}\right)\right]$

$\Rightarrow I = \frac{2}{3}\tan^{-1}\frac{1}{3}$

Q4. Evaluate $\int_1^3 (2x^2 + 5x)$ dx as a limit of a sum. **6 Marks**

Ans:

$\int_1^3 (2x^2 + 5x)dx = \lim_{h\to 0} h[f(1) + f(1+h) + f(1+2h) + \ldots\ldots\ldots + f(1 + \overline{n-1}\,h)]$

where f(x) = $2x^2$ + 5x and h = $\frac{2}{n}$ or nh 2

f(1) = 7

f(1 + h) = 2 $(1 + h)^2$ + 5 (1 + h) = 7 + 9h + $2h^2$

f(1 + 2h) = 2 $(1 + 2h)^2$ + 5 (1 + 2h) = 7 + 18h + 2.2^2h^2

f(1 + 3h) = 2 $(1 + 3h)^2$ + 5 (1 + 3h) = 7 + 27h + 2.3^2h^2

f(1 + (n – 1) h) = 7 + 9 (n – 1) h + 2.$(n-1)^2$ h^2.

$I = \lim_{h\to 0}\left[h[7n + 9h\frac{n(n-1)}{2} + 2h^2\cdot\frac{n(n-1)(2n-1)}{6}\right]$

$= \lim_{h\to0}\left[7nh + \frac{9}{2}nh\ (nh - h) + \frac{1}{3}nh\ (nh - h)(2nh - h)\right]$

$= 14 + 18 + \frac{16}{3} = \frac{112}{3}.$

Q5. Evaluvate the following intregals: **6 Marks**

$\int \frac{1}{p+q\tan x}\,dx$

Ans:

Let $I = \int \frac{1}{p+q\tan x}\,dx$

$= \int \frac{1}{P+\frac{q\sin x}{\cos x}}\,dx$

$= \int \frac{\cos x}{q\sin x+p\cos x}\,dx$

Let $\cos x = A(q\sin x + p\cos x) + B(q\cos x - p\sin x)$

$\Rightarrow \cos x = (Ap + Bq)\cos x + (Aq - Bp)\sin x$

Compairing coefficient of like terms

$Ap + Bq = 0\ldots(1)$

$Aq + Bp = 1\ldots(2)$

Multipiying eq (1) by p and eq (2) by q and then adding

$\Rightarrow Ap^2 + Bpq = p$

$\Rightarrow Aq^2 + Bpq = 0$

$\Rightarrow A = \frac{p}{p^2q^2}$

Putting value of A in eq (1)

$\frac{p^2}{p^2+q^2} + Bq = 1$

$\Rightarrow Bq = 1 - \frac{p^2}{p^2+q^2}$

$\Rightarrow Bq = \frac{p^2+q^2-p^2}{p^2+q^2}$

$\Rightarrow B = \frac{q}{p^2+q^2}$

$\therefore I = \int\left[\frac{p^2}{p^2+q^2} \times \frac{(q\sin x+p\cos x)}{(q\sin x+p\cos x)} + \frac{q}{p^2+q^2} \times \frac{(q\cos x-p\sin x)}{(q\sin x+p\cos x)}\right]dx$

$= \frac{q}{p^2+q^2}\int dx + \frac{p^2}{p^2+q^2}\int\left(\frac{q\cos x-p\sin x}{q\sin x+p\cos x}\right)dx$

Putting $q\sin x + p\cos x = t$

$\Rightarrow (q\cos x - p\sin x)\ dx = dt$

$\therefore I = \frac{p}{q+q^2}\int dx + \frac{q}{p^2+q^2}\int \frac{1}{t}dt$

$= \frac{p}{q+q^2}x + \frac{q}{p^2+q^2}\ln|q\sin x + p\cos x| + C$

Q6. Evaluate the following integrals: **6 Marks**

$\int_{-\pi}^{\pi} \frac{2x(1+\sin x)}{1+\cos^2 x}\,dx$

Ans:

Let $I = \int_{-\pi}^{\pi} \frac{2x(1+\sin x)}{1+\cos^2 x}\,dx$

Then,

$I = \int_{-\pi}^{\pi} \frac{2x}{1+\cos^2 x}\,dx + \int_{-\pi}^{\pi} \frac{2x\sin x}{1+\cos^2 x}\,dx$

$= I_1 + I_2$

Consider $f(x) = \frac{2x}{1+\cos^2 -x}$

Now,

$f(-x) = \frac{2(-x)}{1+\cos^2(\pi-x)} = -\frac{2x}{1+(-\cos x)^2} = -\frac{2x}{1+\cos^2 x} = -f(x)$

$\therefore\ I_1 = \int_{-\pi}^{\pi} \frac{2x}{1+\cos^2 x}\,dx = 0\left[\int_{-a}^{a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, & \text{if } f(-x) = f(x) \\ 0, & \text{if } f(-x) = -f(x)\end{cases}\right]$

Again, consider $g(x) = \frac{2x\sin x}{1+\cos^2 x}$

$g(-x) = \frac{2(-x)\sin(-x)}{1+\cos^2(-x)} = \frac{2x\sin x}{1+\cos^2 x} = g(x)\ [\sin(-x) = -\sin x \text{ and } \cos(-x) = \cos x]$

$\therefore\ I_2 = \int_{-\pi}^{\pi} \frac{2x\sin x}{1+\cos^2 x}\,dx$

$= 2 \times 2 \int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\, dx \left[\int_{-a}^{a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, & \text{if } f(-x) = f(x) \\ 0, & \text{if } f(-x) = -f(x) \end{cases} \right]$

$= 4 \int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\, dx \ldots (i)$

Then,

$I_2 = 4 \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1+\cos^2(\pi - x)}\, dx$

$= 4 \int_0^{\pi} \frac{(\pi - x)\sin x}{1+\cos^2 x}\, dx \ldots . (ii)$

Adding (i) and (ii) we get

$2I_2 = 4 \int_0^{\pi} \frac{\pi \sin x}{1+\cos^2 x}\, dx$

$\Rightarrow 2I_2 = 4 \int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\, dx$

Put $\cos x = z$

$\Rightarrow -\sin x dx = dz$

When $x \to 0,\ z \to 1$

When $x \to \pi,\ z \to -1$

$\therefore\ 2I_2 = -4\pi \int_1^{-1} \frac{dz}{1+z^2}$

$\Rightarrow 2I_2 = -4\pi \times \left[\tan^{-1} z \right]_1^{-1}$

$\Rightarrow 2I_2 = -4\pi \left[\tan^{-1}(-1) - \tan^{-1} 1 \right]$

$\Rightarrow 2I_2 = -4\pi \left(-\frac{\pi}{4} - \frac{\pi}{4} \right) = 2\pi^2$

$\Rightarrow I_2 = \pi^2$

$\therefore\ I = I_1 + I_2$

$\Rightarrow I = 0 + \pi^2 = \pi^2$

Q7. Evaluate the following integrals: **6 Marks**

$\int \frac{\sin^{-1} x}{x^2} dx$

Ans:

Let $I = \int \frac{\sin^{-1} x}{x^2} dx$

$= \int \left(\frac{1}{x^2}\right)(\sin^{-1} x) dx$

$I = \left[\sin^{-1} x \int \frac{1}{x^2} dx - \int \left(\frac{1}{\sqrt{1-x^2}} \int \frac{1}{x^2} dx \right) dx \right]$

$= \sin^{-1} x \left(-\frac{1}{x} \right) - \int \frac{1}{\sqrt{1-x^2}} \left(-\frac{1}{x} \right) dx$

$I = -\frac{1}{x} \sin^{-1} x + \int \frac{1}{x\sqrt{1-x^2}} dx$

$I = -\frac{1}{x} \sin^{-1} x + I_1 \ldots (1)$

Where,

$I_1 = \int \frac{1}{x\sqrt{1-x^2}} dx$

Let $1 - x^2 = t^2$

$-2x\, dx = 2t\, dt$

$I_1 = \int \frac{x}{x^2\sqrt{1-x^2}} dx$

$= -\int \frac{t dt}{(1-t^2)\sqrt{t}}$

$= -\int \frac{dt}{(1-t^2)}$

$= \int \frac{1}{t^2-1} dt$

$= \frac{1}{2} \log \left| \frac{t-1}{t+1} \right|$

$= \frac{1}{2} \log \left| \frac{\sqrt{1-x^2}-1}{\sqrt{1-x^2}+1} \right| + C_1$

Now

$I = -\frac{\sin^{-1} x}{x} + \frac{1}{2} \log \left| \left(\frac{\sqrt{1-x^2}-1}{\sqrt{1-x^2}+1} \right) \left(\frac{\sqrt{1-x^2}-1}{\sqrt{1-x^2}-1} \right) \right| + C$

$= -\frac{\sin^{-1} x}{x} + \frac{1}{2} \log \left| \frac{(\sqrt{1-x^2}-1)^2}{1-x^2-1} \right| + C$

$= -\frac{\sin^{-1} x}{x} + \frac{1}{2} \log \left| \frac{(\sqrt{1-x^2}-1)^2}{-x^2} \right| + C$

$= -\frac{\sin^{-1}x}{x} + \log\left|\frac{\sqrt{1-x^2}-1}{-x}\right| + C$

$I = -\frac{\sin^{-1}x}{x} + \log\left|\frac{1-\sqrt{1-x}^2}{x}\right| + C$

Q8. Find the integrals of the functions in Exercises: **6 Marks**

$\sin^{-1}(\cos x)$

Ans:

$\sin^{-1}(\cos x)$

Let $\cos x = t$

Then, $\sin x = \sqrt{1-t^2}$

$\Rightarrow (-\sin x)dx = dt$

$dx = \frac{-dt}{\sin x}$

$dx = \frac{-dt}{\sqrt{1-t^2}}$

$\therefore \int \sin^{-1}(\cos x)dx = \int \sin^{-1} t\left(\frac{-dt}{\sqrt{1+t^2}}\right)$

$= -\int \frac{\sin^{-1} t}{\sqrt{1-t^2}}dt$

Let $\sin^{-1} t = u$

$\Rightarrow \frac{1}{\sqrt{1-t^2}}dt = du$

$\therefore \int \sin^{-1}(\cos x)dx = \int u\, du$

$= -\frac{u^2}{2} + C$

$= \frac{-(\sin^{-1} t)^2}{2} + C$

$= \frac{-\left[\sin^{-1}(\cos x)\right]^2}{2} + C \quad \ldots(1)$

It is known that,

$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$

$\therefore \sin^{-1}(\cos x) = \frac{\pi}{2} - \cos^{-1}(\cos x) = \left(\frac{\pi}{2} - x\right)$

Substituting in equation(1),we obtain

$\int \sin^{-1}(\cos x)\, dx = \frac{-\left[\frac{\pi}{2}-x\right]^2}{2} + C$

$= -\frac{1}{2}\left(\frac{\pi^2}{2} + x^2 - \pi x\right) + C$

$= -\frac{\pi^2}{8} - \frac{x^2}{2} + \frac{1}{2}\pi x + C$

$= \frac{\pi x}{2} - \frac{x^2}{2} + \left(C - \frac{\pi^2}{8}\right)$

$= \frac{\pi x}{2} - \frac{x^2}{2} + C$

Q9. $\int \frac{x}{\sqrt{x+a}-\sqrt{x+b}}dx$ **6 Marks**

Ans:

$\int \frac{x}{\sqrt{x+a}-\sqrt{x+b}}dx$

$= \int \frac{x}{\sqrt{x+a}-\sqrt{x+b}} \times \frac{\sqrt{x+a}+\sqrt{x+b}}{\sqrt{x+a}+\sqrt{x+b}}dx$

$= \int \frac{x(\sqrt{x+a}+\sqrt{x+b})}{(\sqrt{x+a})^2-(\sqrt{x+b})^2}dx$

$= \int \frac{x(\sqrt{x+a}+\sqrt{x+b})}{x+a-x-b}dx$

$= \frac{1}{a-b}\int x(\sqrt{x+a} + \sqrt{x+b})dx$

$= \frac{1}{a-b}\left[\int x(\sqrt{x+a})dx + \int x(\sqrt{x+b})dx\right]$

$= \frac{1}{a-b}\left[\int(x+a-a)(\sqrt{x+a})dx + \int(x+b-b)(\sqrt{x+b})dx\right]$

$= \frac{1}{a-b}\left[\int(x+a)(\sqrt{x+a})dx - a\int(\sqrt{x+a})dx\right.$

$\left.+ \int(x+b)(\sqrt{x+b})dx - b\int(\sqrt{x+b})dx\right]$

$= \frac{1}{a-b}\left[\int(x+a)^{\frac{3}{2}}dx - a\int(x+a)^{\frac{1}{2}}dx + \int(x+b)^{\frac{3}{2}}dx - b\int(x+b)^{\frac{1}{2}}dx\right]$

$= \frac{1}{a-b}\left[\frac{(x+a)^{\frac{5}{2}}}{\frac{5}{2}} - a\frac{(x+a)^{\frac{3}{2}}}{\frac{3}{2}} + \frac{(x+b)^{\frac{5}{2}}}{\frac{5}{2}} - b\frac{(x+b)^{\frac{3}{2}}}{\frac{3}{2}}\right.$ + c where, c is an arbitrary constant.

$= \frac{1}{a-b}\left[\frac{2}{5}(x+a)^{\frac{5}{2}} - \frac{2a}{3}(x+a)^{\frac{3}{2}} + \frac{2}{5}(x+b)^{\frac{5}{2}} - \frac{2b}{3}(x+b)^{\frac{3}{2}}\right]$ + c where, c is an arbitrary constant.

Hence, $\int \frac{x}{\sqrt{x+a}-\sqrt{x+b}} dx = \frac{1}{a-b}\left[\frac{2}{5}(x+a)^{\frac{5}{2}} - \frac{2a}{3}(x+a)^{\frac{3}{2}} + \frac{2}{5}(x+b)^{\frac{5}{2}} - \frac{2b}{3}(x+b)^{\frac{3}{2}}\right]$ + c where, c is an arbitrary constant.

Q10. Evaluate the following intregals: **6 Marks**

$\int \frac{x+2}{\sqrt{x^2-1}} dx$

Ans:

Let $x+2 = A\frac{d}{dx}(x^2-1) + B \ldots (1)$

$\Rightarrow x+2 = A(2x) + B$

Equating the coefficient of x and constant term on both sides, we obtain

$2A = 1 \Rightarrow A = \frac{1}{2}$

$B = 2$

From (1), we obtain

$(x+2) = \frac{1}{2}(2x) + 2$

Then, $\int \frac{x+2}{\sqrt{x^2-1}} dx = \int \frac{\frac{1}{2}(2x)+2}{\sqrt{x^2-1}} dx$

$= \frac{1}{2}\int \frac{2x}{\sqrt{x^2-1}} dx + \int \frac{2}{\sqrt{x^2-1}} dx \ldots (2)$

$\text{In } \frac{1}{2}\int \frac{2x}{\sqrt{x^2-1}} dx,$

Let $x^2 - 1 = t \Rightarrow 2x\, dx = dt$

In $\frac{1}{2}\int \frac{2x}{\sqrt{x^2-1}} dx$

let $x^2 - 1 = t \Rightarrow 2x\, dx = dt$

$\frac{1}{2}\int \frac{2x}{\sqrt{x^2-1}} dx = \frac{1}{2}\int \frac{dt}{\sqrt{t}}$

$= \frac{1}{2}[2\sqrt{t}]$

$= \sqrt{t}$

$= \sqrt{x^2-1}$

Then, $\int \frac{2}{\sqrt{x^2-1}} dx = 2\int \frac{1}{\sqrt{x^2-1}} dx = 2\log\left|x + \sqrt{x\ +1}\right|$

From equation (2), we obtain

$\int \frac{x+2}{\sqrt{x^2-1}} dx = \sqrt{x^2-1} + 2\log\left|x + \sqrt{x^2-1}\right| + C$

Q11. **6 Marks**

Evaluate: $\int \frac{x \sin^{-1} x}{\sqrt{x\text{-}1^2}} dx.$

Ans:

$I = \int \frac{x \sin^{-1} x}{\sqrt{1\text{ - }x^2}} dx$; Let x = sinθ ⇒ dx = cosθ dθ

$= \int \frac{\sin\theta \cdot \theta}{\cos\theta} \cos\theta\, d\theta = \int \theta \sin\theta\, d\theta$

= –θ cosθ + ∫cosθ dθ = –θ cosθ + sinθ + c

$= -\sin^{-1} x(\sqrt{1\text{-}x^2}) + x + c.$

Q12. Evaluate: **6 Marks**

$\int \frac{1}{\cos^4 x + \sin^4 x} dx$

Ans:

$I = \int \frac{1}{\cos^4 x + \sin^4 x} dx$

dividing numerator and denominator by $\cos^4 x$

$= \int \frac{\sec^4 x}{1+\tan^4 x} dx = \int \frac{(1+\tan^2 x)\sec^2 x}{1+\tan^4 x} dx$

Putting tan x = t

$\therefore \sec^2 x\, dx = dt$

$= \int \frac{(t^2+1)dt}{t^4+1} = \int \frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}} dt \left\{ \text{dividing by } t^2 \right\}$

$= \int \frac{dz}{z^2+(\sqrt{2})^2} \text{ where } t - \frac{1}{t} = z$

$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{z}{\sqrt{2}}\right) + c$

$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t^2-1}{\sqrt{2}t}\right) + c = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan^2 x - 1}{\sqrt{2}\tan x}\right) + c.$

Q13. Integrate the function in Exercise: **6 Marks**

$(x^2+1)\log x$

Ans:

Let $\mathrm{I}=\int(\mathrm{x}^2+1)\log \mathrm{x}\ \mathrm{dx}=\int \mathrm{x}^2\log \mathrm{x}\ \mathrm{dx}+\int \log \mathrm{x}\ \mathrm{dx}$

Let $I = I_1 + I_2$....(1)

where, $\mathrm{I}_1=\int \mathrm{x}^2\log \mathrm{x}\ \mathrm{dx}$ and $\mathrm{I}_2=\int \log\mathrm{x}\ \mathrm{dx}$

$\mathrm{I}_1=\int \mathrm{x}^2\log \mathrm{x}\ \mathrm{dx}$

Taking log x as first function and x^2 as secound function and integrating by parts, we obtain

$\int \mathrm{I.\,II}\ \mathrm{dx}=\mathrm{I}\int \mathrm{II}\ \mathrm{dx}-\int\left\{\frac{\mathrm{d}}{\mathrm{dx}}\mathrm{I}\int \mathrm{II}\ \mathrm{dx}\right\}\mathrm{dx}$

$\mathrm{I}_1=\log \mathrm{x}\int \mathrm{x}^2\mathrm{dx}-\int\left\{\left(\frac{\mathrm{d}}{\mathrm{dx}}\log \mathrm{x}\right)\int \mathrm{x}^2\mathrm{dx}\right\}\mathrm{dx}$

$=\log \mathrm{x}.\,\frac{\mathrm{x}^3}{3}-\int\frac{1}{\mathrm{x}}.\,\frac{\mathrm{x}^3}{3}\mathrm{dx}$

$=\frac{\mathrm{x}^3}{3}\log \mathrm{x}-\frac{1}{3}\left(\int \mathrm{x}^2\mathrm{dx}\right)$

$=\frac{\mathrm{x}^3}{3}\log \mathrm{x}-\frac{\mathrm{x}^3}{9}+\mathrm{C}\ldots(2)$

$\mathrm{I}_2=\int \log \mathrm{x}\ \mathrm{dx}$

Taking log x as first function and 1 as secound function and integrating by parts, we obtain

$\mathrm{I}_2=\log \mathrm{x}\int 1.\mathrm{dx}-\int\left\{\left(\frac{\mathrm{d}}{\mathrm{dx}}\log \mathrm{x}\right)\int 1.\mathrm{dx}\right\}\mathrm{dx}$

$=\log \mathrm{x}.\,\mathrm{x}-\int\frac{1}{\mathrm{x}}.\,\mathrm{x}\ \mathrm{dx}$

$=\mathrm{x}\log \mathrm{x}-\int 1\mathrm{dx}$

$=\mathrm{x}\log \mathrm{x}-\mathrm{x}+\mathrm{C}_2\ldots(3)$

Using equations (2) and (3) in (1), we obtain

$\mathrm{I}=\frac{\mathrm{x}^3}{3}\log \mathrm{x}-\frac{\mathrm{x}^3}{9}+\mathrm{C}_1+\mathrm{x}\log\mathrm{x}-\mathrm{x}+\mathrm{C}_2$

$=\frac{\mathrm{x}^3}{3}\log \mathrm{x}-\frac{\mathrm{x}^3}{9}+\mathrm{x}\log\mathrm{x}-\mathrm{x}+(\mathrm{C}_1+\mathrm{C}_2)$

$=\left(\frac{\mathrm{x}^3}{3}+\mathrm{x}\right)\log \mathrm{x}-\frac{\mathrm{x}^3}{9}-\mathrm{x}+\mathrm{C}$

Q14. Evaluate $\int_0^2(x^2+2x+1)\mathrm{dx}$ as limit of sums. **6 Marks**

Ans:

Here $\mathrm{h}=\frac{2-0}{n}=\frac{2}{n},\mathrm{f}\,(\mathrm{x})=x^2+2x+1$

$\therefore \mathrm{I}=\lim_{h\to0}\left[\mathrm{f}\,(0)+\mathrm{f}\,(\mathrm{h})+\mathrm{f}\,(2\mathrm{h})+\ldots\ldots\ldots\ldots+\mathrm{f}\overline{(\mathrm{n}-1)\mathrm{h}}\right]$

$=\lim_{h\to0}\mathrm{h}\left[1+(h^2+2h+1)+(2^2h^2+2.2h+1)+\ldots\ldots\ldots(n-1^2)h^2+2(n-1)h+1)\right]$

$=\lim_{\substack{n\to\infty\\h\to0}}\frac{2}{n}\left[(1+1+1+1+1+\ \ldots\ldots\ldots)+\mathrm{h}^2\left(1+2^2+3^2+\ \ldots.\right)\right]$

$=\lim_{n\to\infty}\frac{2}{n}\left[n+\frac{4}{n^2}\frac{(n-1)n(2n-1)}{6}+\frac{4}{n}.\,\frac{(n-1)n}{n}\right]$

$=\lim_{n\to\infty}\left[2+\frac{8}{6}\frac{n-1}{n}\frac{2n-1}{n}+4\frac{n-1}{n}\right]$

$=2+4+\frac{8}{3}=\frac{26}{3}$

Q15. Integrate the rational function in exercise: **6 Marks**

$\frac{\mathrm{x}}{(\mathrm{x}^2+1)(\mathrm{x}-1)}$

Ans:

$\frac{\mathrm{x}}{(\mathrm{x}^2+1)(\mathrm{x}-1)}=\frac{\mathrm{Ax}+\mathrm{B}}{\mathrm{x}^2+1}+\frac{\mathrm{C}}{\mathrm{x}-1}\ldots(\mathrm{i})$

⇒ x = (Ax + B) (x - 1) + C(x^2 + 1)

⇒ x = Ax^2 - Ax + Bx - B + Cx^2 + C

Comparing coefficients of x^2, A + C = 0 ....(ii)

Comparing coefficients of x, –A + B = 1 ....(iii)

Comparing constant terms, –B + C = 0 ....(iv)

Solving eq. (ii), (iii) and (iv), we get

$\mathrm{A}=\frac{-1}{2},\ \mathrm{B}=\frac{1}{2}$ and $\mathrm{C}=\frac{1}{2}$

Putting the values of A, B and C in eq. (i),

$\frac{\mathrm{x}}{(\mathrm{x}^2+1)(\mathrm{x}-1)}=\frac{\frac{-1}{2}\mathrm{x}+\frac{1}{2}}{\mathrm{x}^2+1}+\frac{\frac{1}{2}}{\mathrm{x}-1}$

$\Rightarrow\ \frac{\mathrm{x}}{(\mathrm{x}^2+1)}=\frac{-1}{2}.\,\frac{\mathrm{x}}{\mathrm{x}^2+1}+\frac{1}{2}.\,\frac{1}{\mathrm{x}^2+1}+\frac{1}{2}.\,\frac{1}{\mathrm{x}-1}=\frac{-1}{4}.\,\frac{2\mathrm{x}}{\mathrm{x}^2+1}+\frac{1}{2}.\,\frac{1}{\mathrm{x}^2+1}+\frac{1}{2}.\,\frac{1}{\mathrm{x}-1}$

$\Rightarrow \frac{x}{(x^2+1)(x-1)}dx = \frac{-1}{4}\int \frac{2x}{x^2+1}dx + \frac{1}{2}\int \frac{1}{x^2+1}dx + \frac{1}{2}\int \frac{1}{x-1}dx$

$\Rightarrow \frac{x}{(x^2+1)(x-1)}dx = \frac{-1}{4}\log|x^2+1| + \frac{1}{2}\tan^{-1}x + \frac{1}{2}\log|x-1| + c$

$\Rightarrow \frac{x}{(x^2+1)(x-1)} = \frac{-1}{4}\log(x^2+1) + \frac{1}{2}\tan^{-1}x + \frac{1}{2}\log|x-1| + c$

Q16. Evaluate: $\int_{-a}^{a} \sqrt{\frac{a-x}{a+x}}dx$ **6 Marks**

Ans:

$I = \int_{-a}^{a} \sqrt{\frac{a-x}{a+x}}dx = \int_{-a}^{a} \frac{a-x}{\sqrt{a^2-x^2}}dx = \int_{-a}^{a} \frac{xdx}{\sqrt{a^2-x^2}}$

I_1 is even function and I_2 is odd function

$\therefore I_2 = 0$

$i = 2a\int_{-a}^{a} \frac{dx}{\sqrt{a^2-x^2}} = 2a.\frac{\pi}{2} = \pi a$

$\therefore I = \pi a$

Q17. Evaluate the following integrals: **6 Marks**

$\int \frac{(3\sin x-2)\cos x}{5-\cos^2 x-4\sin x}\ dx$

Ans:

Let $I = \int \frac{(3\sin x-2)\cos x}{5-\cos^2 x-4\sin x}\ dx$

$\therefore\ I = \int \frac{(3\sin x-2)\cos x}{5-(1-\sin^2 x)-4\sin x}\ dx$

$\Rightarrow I = \int \frac{(3\sin x-2)\cos x}{5-1+\sin^2 x-4\sin x}\ dx$

Substitute $\sin x = t$

$\Rightarrow \cos x\ dx = dt$

Thus,

$I = \int \frac{(3t-2)}{4+t^2-4t}\ dt$

$I = \int \frac{(3t-2)}{t^2-4t+4}\ dt$

$I = \int \frac{(3t-2)}{(t-2)^2}\ dt$

Now let us separate the integrand into the simplest form using partial fractions.

$\frac{(3t-2)}{(t-2)^2} = \frac{A}{(t-2)} + \frac{B}{(t-2)^2}$

$= \frac{A(t-2)+B}{(t-2)^2}$

$= \frac{At-2A+B}{(t-2)^2}$

$\Rightarrow 3t-2 = At-2A+B$

Comparing the coefficients, we have,

$A = 3$

and

$-2A+B = -2$

Substituting the value of A = 3 in the above equation, we have,

$\Rightarrow -2\times 3+B = -2$

$\Rightarrow -6+B = -2$

$\Rightarrow B = 6-2$

$\Rightarrow B = 4$

Thus, $I = \int \frac{(3t-2)}{(t-2)^2}\ dt$ becomes,

$I = \int \frac{3}{(t-2)^2}\ dt + \int \frac{4}{(t-2)^2}\ dt$

$= 3\log|t-2| - 4\left(\frac{1}{t-2}\right) + C$

$= 3\log|2-t| + 4\left(\frac{1}{2-t}\right) + C$

Now, substituting $t = \sin x$, we have,

$= 3\log|2-\sin x| + 4\left(\frac{1}{2-\sin x}\right) + C$

Q18. Integrate the function in Exercise: **6 Marks**

$\frac{\sin^{-1}\sqrt{x}-\cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x}+\cos^{-1}\sqrt{x}}, x \in [0,1]$

Ans:

we know that $\sin^{-1}\sqrt{x}+\cos\sqrt{x}=\frac{\pi}{2}\Rightarrow\cos^{-1}\sqrt{x}=\frac{\pi}{2}-\sin^{-1}\sqrt{x}$

$\therefore I=\int\frac{\sin^{-1}\sqrt{x}\left(\frac{\pi}{2}-\sin^{-1}\sqrt{x}\right)}{\frac{\pi}{2}}dx=\frac{2}{\pi}\int\left(2\sin^{-1}\sqrt{x}-\frac{\pi}{2}\right)dx$

$\Rightarrow I=\frac{4}{\pi}\int\sin^{-1}\sqrt{x}dx-\int 1dx=\frac{4}{\pi}\int\sin^{-1}\sqrt{x}\,dx-x+c$

putting $\sqrt{x}=\sin\theta\Rightarrow x=\sin^2\theta\Rightarrow dx=2\sin\theta\cos\theta d\theta=\sin 2\theta\,d\theta$

$\therefore I=\frac{4}{\pi}\int\left(\sin^{-1}(\sin\theta).\sin 2\theta\right)d\theta-x+c=\frac{4}{\pi}\int(\theta.\sin 2\theta)d\theta-x+c$

Q19. Evaluate the following integrals: **6 Marks**

$\int\frac{1}{\sin x+\sin 2x}\,dx$

Ans:

Let $I=\int\frac{1}{\sin x+\sin 2x}\,dx$

$=\int\frac{dx}{\sin x+2\sin x\cos x}$

$=\int\frac{\sin x\,dx}{(1-\cos^2 x)+2(1-\cos^2 x)\cos x}$

Let $\cos x=t\Rightarrow-\sin x\,dx=dt$

$\therefore I=\int\frac{dt}{(t^2-1)+2(t^2-1)t}$

$=\int\frac{dt}{(t^2-1)(1+2t)}$

Let $\int\frac{1}{(t^2-1)(1+2t)}=\frac{A}{t-1}+\frac{B}{t+1}+\frac{C}{1+2t}$

Put t = 1

⇒ 1 = 6A ⇒ $A=\frac{1}{6}$

Put t = -1

⇒ 1 = 2B ⇒ $B=\frac{1}{2}$

put $t=-\frac{1}{2}$

$\Rightarrow 1=-\frac{3}{4}C\Rightarrow C=-\frac{4}{3}$

Thus,

$I=\frac{1}{6}\int\frac{dt}{t-1}+\frac{1}{2}\int\frac{dt}{t+1}-\frac{4}{3}\int\frac{dt}{1+2t}$

$=\frac{1}{6}\log|t-1|+\frac{1}{2}\log|t+1|-\frac{2}{3}\log|1+2t|+C$

Hence,

$I=\frac{1}{6}\log|\cos x-1|+\frac{1}{2}\log|\cos x+1|$

$-\frac{2}{3}\log|1+2\cos x|+C$

Q20. Evaluate: **6 Marks**

$\int_{\pi/6}^{\pi/3}\frac{dx}{1+\sqrt{\tan x}}.$

Ans:

$I=\int_{\pi/6}^{\pi/3}\frac{dx}{1+\sqrt{\tan x}}=\int_{\pi/6}^{\pi/3}\frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}dx\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(i)$

$x\to\left(\pi/3+\pi/6-x\right)$

$=\int_{\pi/6}^{\pi/3}\frac{\sqrt{\cos(\pi/2-x)}}{\sqrt{\cos(\pi/2-x)}+\sqrt{\sin(\pi/2-x)}}dx=\int_{\pi/6}^{\pi/3}\frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx\ldots\ldots\ldots\ldots(ii)$

Adding (i) and (ii) to get

$2I=\int_{\pi/6}^{\pi/3}1.\,dx=[x]_{\pi/6}^{\pi/3}=\pi/3-\pi/6=\pi/6$

$\Rightarrow I=\pi/12.$